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Anomalous D-brane charge in F-theory compactifications

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Abstract: Tadpole cancellation in F-theory on an elliptic Calabi-Yau fourfold $X \rightarrow B_3$ demands some spacetime-filling three-branes (points in B_3). If moved to the discriminant surface, which supports the gauge group, and dissolved into a finite size instanton, the second Chern class of the corresponding bundle E is expected to give a compensating contribution. However the dependence of D-brane charge on the geometry of W and on the embedding $i : W \rightarrow B_3$ gives a correction to $c_2(E)$. We show how this is reconciled by considering the torsion sheaf i_*E and discuss some integrality issues related to global properties of X as well as the moduli space of this object.

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Two well-studied models for $N = 1$ string compactifications are the heterotic string on a Calabi-Yau three-fold with a vector bundle embedded in $E_8 \times E_8$ and F-theory on a Calabi-Yau four-fold X (in the following when referring to cohomological formulas involving X or couplings given by integration over X, we denote by X the smooth resolved four-fold) [1, 2, 3]. In the last case X is an elliptic fibration over B_3 which in turn is a $\mathbf{P}^1$ fibration[1] over B_2 in such a way that X is $K3$ fibered over B_2. Both of these models require in general for consistency the inclusion of some brane-impurities: the heterotic string leads to some five-branes because of anomaly cancellation [4], and the F-theory vacuum leads to a number of space-time filling three-branes because of tadpole cancellation [5]. These three-branes are located at points on the base B_3.

Let us briefly recall the possible contributions to the tadpole equation. The first one comes from the coupling [6]

$$-\int_{\mathbf{R}^3 \times X} C \wedge I_8 \,. \tag{1}$$

Here we write the term on the level of M-theory. Corresponding terms involving a coupling of B^{NS} with an 8-form in curvature in type IIA give rise to a 2D term [7]. Lifting of such a term to F-theory is also possible [8], and gives rise to the above-mentioned tadpole. Indeed using $\int_X I_8 = \chi(X)/24$ [9], one finds for the number of three-branes

$$n_3 = \frac{\chi(X)}{24} \,. \tag{2}$$

Similarly when one takes into account non-vanishing four-fluxes, the Chern-Simons coupling

$$\int_{\mathbf{R}^3 \times X} C \wedge G \wedge G \tag{3}$$

also contributes to the number of three-branes [10] ($G = \frac{1}{2\pi} dC$)

$$\frac{\chi(X)}{24} = n_3 + \frac{1}{2} \int_X G \wedge G \,. \tag{4}$$

[1]The fibration structure of $\pi : B_3 \to B_2$ is described [4] by assuming the $\mathbf{P}^1$ bundle over B_2 (with section r) to be a projectivization of a vector bundle $Y = \mathcal{O} \oplus \mathcal{T}$, where $\mathcal{T}$ is a line bundle over B_2 and the cohomology class $t = c_1(\mathcal{T})$ encodes the $\mathbf{P^1}$ fibration structure. Let N denote the normal bundle of W in B_3 with $c_1(N) = -t$ from $rr = -tr$ (on the heterotic side t indicates an asymmetry between the cohomological data of the two bundles [4]). To explain this relation and also for later use let us look into the relevant geometry. Let $\mathcal{O}(1)$ be the line bundle on the total space of $\mathbf{P}(Y) \to B_2$ which is fibrewise the usual $\mathcal{O}(1)$. Let a, b be homogeneous coordinates of $\mathbf{P}(Y)$ and think of a, b as sections, respectively, of $\mathcal{O}(1)$ and $\mathcal{O}(1) \otimes \mathcal{T}$ over B_2 with Chern-classes $r = c_1(\mathcal{O}(1))$ and $r + t = c_1(\mathcal{O}(1) \otimes \mathcal{T})$. Then the cohomology ring of B_3 is generated over the cohomology ring of B_2 by r with the relation $r(r+t) = 0$ (meaning that the divisors a, b which are dual to r resp. $r + t$ do not intersect). The Chern-classes of B_3 are then computed by applying the adjunction formula $c(B_3) = (1 + c_1 + c_2)(1 + r)(1 + r + t)$ (here unspecified Chern classes refer to B_2: $c_i = c_i(B_2) = \pi^*(c_i(B_2))$).

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Here we consider the F-theory analog of M-theory four-flux in the limit that the area of the elliptic fibers is very small. The four form G on X can be expanded in terms of forms denoted by p_2, H_3 and g_4 (subscripts indicating the degree) [11]

$$G = p_2 \wedge \chi + \sum_i H_3^i \wedge \theta^i + g_4 \,, \tag{5}$$

where χ is an integral two-form generating the two-dimensional cohomology of the fibers and θ^i, $i = 1, 2$, is a basis of $H^1(T^2)$. Setting $g = p = 0$ guarantees that G is part of the primitive $H^{2,2}(X)$ cohomology and is odd under the fiber involution. In type-IIB theory H can be expressed in terms of the NS and Ramond three-form field strength $H^{\rm NS}, H^R$ and chosen to be an integral $(2,1)$ form, well defined up to $\mathrm{Sl}_2(\mathbb{Z})$ transformations around the seven-brane loci. So $G \wedge G$ goes in the F-theory/type-IIB limit to[2]

$$\frac{1}{2}\int_X G \wedge G \longrightarrow \int_{B_3} \frac{1}{\tau_2} H \wedge \overline{H} \tag{6}$$

with $H = H^R - \tau H^{\rm NS}$ and $\overline{H} = H^R - \overline{\tau} H^{\rm NS}$.

In case the point p (which constitutes the compact part of the world-volume of a three-brane) lies not only in B_3 but actually in W, one can consider p as a small instanton of an unbroken gauge group located in the (compact part W of the) corresponding seven-brane world-volume, which corresponds to a component of the discriminant locus of the elliptic fibration. The discriminant locus in general decomposes into components (resp. seven-branes); for example matching the perturbative heterotic gauge content requires having three components, two of which carry the (unbroken heterotic) gauge group $G_1 \times G_2$, and the third one — I_1 singular fibers. Here we will think of a situation where we have only two seven-branes with G_1 over W and I_1 singularity located over D_1, corresponding to a heterotic bundle (V_1, E_8) [13].

Note that in general the seven-branes are intersecting. In our context this means that the component W of the discriminant which carries the G_1 singularity will intersect the component carrying the I_1 singularity. However, since only on W a Chan-Paton bundle E is switched on (the group of the I_1 surface is $\mathrm{U}(1)$) at the moment we do not have to worry about intersections (but see remark 2).

One can also have an instanton in a rank $\mathrm{rk}(E)$ bundle E over W, breaking part of the F-theory gauge group G associated with W [3]. Then the possibility of transitions where such an instanton becomes point-like and leaves the (part of the) discriminant surface W in B_3 arises, leading to a further modification of the consistency condition [3]

$$\frac{\chi(X)}{24} = n_3 + \int_W c_2(E) + \int_{B_3} \frac{1}{\tau_2} H \wedge \overline{H} \,. \tag{7}$$

[2]We will not discuss here the special case where the four-flux is localized around fibers over W and it lifts to the field strength of the corresponding gauge field (cf. [12]) with $G \wedge G$ leading to the term $c_2(E)$.

The last possibility fits with the general philosophy that one always has in connection with a D-brane a vector bundle over its world-volume. Moreover, one can nicely describe transitions associated with small instantons using this formula, balancing the instanton number with the number of three-branes.

However, it is known that the D-brane charge is given in terms of the topology of the embedded worldvolume W ($i : W \hookrightarrow B = B_3$) [14] and the topology of E

$$q = \mathrm{ch}(i_! E)\sqrt{\hat{A}(B)} \tag{8}$$

and thus in general one has to take into account in (7) the non-trivial effects associated with the embedding i. Then a general contribution to the tadpole is no longer just given by integrating, as in $\int_W c_2(E)$, the top class of the bundle over the compact part of the world-volume, but should be deduced from the D3-brane coupling to D7 and is of the form

$$-\int_{\mathbf{R^4}} A_4 \int_W Y_4\,. \tag{9}$$

Here Y_4 denotes the degree four part of the anomalous coupling Y given by [15, 16, 14, 17]

$$Y = \mathrm{ch}(E) e^{-\frac{1}{2}c_1(N)}\sqrt{\frac{\hat{A}(W)}{\hat{A}(N)}} \tag{10}$$

(with the A-roof genus given by $\hat{A} = 1 - p_1/24 + \cdots$ and the Chern character $\mathrm{ch} = \mathrm{rk} + c_1 + c_1^2/2 - c_2 + \frac{c_1^3 - 3c_1c_2 + 3c_3}{6} + \cdots$). Note that one derives (8) by application of Grothendieck-Riemann-Roch theorem to the anomalous couplings of RR fields to Y on D-brane worldvolume.

In summary, because of the anomalous D-brane charge, the contribution $\int_W c_2(E)$ has to be replaced by

$$-\int_W \left(\mathrm{rk}(E)\left(\frac{c_1(N)^2}{8} - \frac{p_1(W)}{48} + \frac{p_1(N)}{48}\right) - \frac{c_1(E)c_1(N)}{2} + \mathrm{ch}_2(E)\right). \tag{11}$$

Since we are concerned here only with sevenbranes, $p_1(N) = c_1(N)^2$ for the line bundle N. With these corrections (7) becomes

$$\frac{\chi(X)}{24} = n_3 - \int_W \left(\mathrm{ch}_2(E) + \mathrm{rk}(E)\left(\frac{c_1(N)^2}{8} + \frac{c_1(N)^2}{48} - \frac{\sigma(W)}{16}\right) - \frac{c_1(E)c_1(N)}{2}\right) + \\ + \int_{B_3} \frac{1}{\tau_2} H \wedge \overline{H}\,. \tag{12}$$

As noted already there are different groups of (intersecting) sevenbranes, and one has to sum over the "instanton" contributions from all of them (including those with trivial gauge bundles), so a sum is assumed in the second term on the right-hand side.

In order to find an interpretation of (12), we once more use the Grothendieck-Riemann-Roch theorem (see [18]) for a holomorphic map $i : W \to B_3$, which in our notation reads (with $\mathrm{Td} = \hat{A}e^{\frac{1}{2}c_1}$)

$$i_*\Big(\mathrm{ch}(E)\,\mathrm{Td}(W)\Big) = \mathrm{ch}(i_!E)\,\mathrm{Td}(B_3)\,, \tag{13}$$

where $i_!$ is the K-theoretic Gysin map (a homomorphism $K(W) \to K(B_3)$) and can be defined as

$$i_!E := \sum_q (-1)^q i_*^q E\,, \tag{14}$$

where $i_*^q E$ are direct image sheaves (the q-th direct image of E). Since $i : W \to B_3$ is an embedding we have $i_*^q E = 0$ for $q > 0$ and $H^p(W, E) \cong H^p(B_3, i_*^0 E)$ for $p \geq 0$ (cf. [18]). Then (14) simplifies for our case to $i_!E = i_*E$ and we have to deal only with the torsion sheaf i_*E.

For further application to (12), we use Gysin homomorphism i_*^c for cohomology which maps classes of codimension p in $B_2 = W$ into classes of codimension p in B_3. Denoting the Poincaré duality on B_3 by $D_{B_3} : H^p(B_3) \to H_{6-p}(B_3)$ (and similarly on W, $D_W : H^p(W) \to H_{4-p}(W)$), we can define the action of i_* on $H^*(W, \mathbb{Q})$ as $i_*^c = D_{B_3}^{-1} i_*^h D_W$ where i_*^h is the usual map induced on homology. It is easy to see that in our case $i_*^c : H^p(W, \mathbb{Q}) \to H^{p+2}(B_3, \mathbb{Q})$ (for example $i_*^c 1 = r$ where r as above denotes the class of W in B_3). If $i : W \hookrightarrow B_3$ is an embedding of W as a submanifold of B_3, $\mathrm{Td}(W) = (\mathrm{Td}(N))^{-1} i^* \mathrm{Td}(B_3)$. We can use now that for $\phi \in H^*(W, \mathbb{Q})$ and $\eta \in H^*(B_3, \mathbb{Q})$, $i_*^c(\phi \wedge i^*\eta) = i_*^c\phi \wedge \eta$ and see that (13) implies the Riemann-Roch theorem for an embedding

$$\mathrm{ch}(i_!E) = i_*^c\Big(\mathrm{ch}(E)(\mathrm{Td}(N))^{-1}\Big)\,, \tag{15}$$

expansion of (15) gives for the Chern characters of the torsion sheaf i_*E

$$\begin{aligned}
\mathrm{ch}_1(i_*E) &= \mathrm{rk}(E)r\,, \qquad \mathrm{ch}_2(i_*E) = i_*^c\left(c_1(E) - \mathrm{rk}(E)\frac{c_1(N)}{2}\right), \\
\mathrm{ch}_3(i_*E) &= i_*^c\left(\mathrm{ch}_2(E) + \mathrm{rk}(E)\left(\frac{c_1(N)^2}{8} + \frac{c_1(N)^2}{24}\right) - c_1(E)\frac{c_1(N)}{2}\right).
\end{aligned} \tag{16}$$

Comparison to (12) shows that the changes in the tadpole equation can be compactly written as

$$\int_W c_2(E) \longrightarrow -\int_B \mathrm{ch}_3(i_*E) + \frac{1}{48}\,\mathrm{rk}(E)\int_B r \wedge p_1(B) \tag{17}$$

and the general condition for the tadpole cancellation in the case of non-trivial D-brane embeddings as (once more with a sum over all brane contributions is assumed)

$$\frac{\chi(X)}{24} = n_3 - \int_{B_3} \mathrm{ch}(i_*E)\sqrt{\hat{A}(B)} + \int_{B_3} \frac{1}{\tau_2} H \wedge \overline{H}\,. \tag{18}$$

Of course one could have arrived at (18) directly from (10), however we have chosen a lengthier presentation shown above since we will need some of the explicit formulae such as (12) and (17) for future use. We see that the original instanton charge $c_2(E)$ [3] is replaced by $[\mathrm{ch}(i_*E)\sqrt{\hat{A}(B)}]_3$ (or as explained in [14], $i_*E \in K(B)$). Note that in the spirit of the F-theory (or better type IIB) reinterpretation of the former M-theory relation and also in the spirit of K-theory interpretation of D-branes [14, 19], the right-hand side is now expressed completely on terms of the base B_3 visible to type IIB (and not in terms of X^4 or W); the left hand side which seems still to involve the complete four-fold not visible to type IIB can also be expressed in terms of data visible to type IIB using the stratification of singularities [13, 5].

Finally let us make some remarks on the modification (18) of the tadpole equation concerning the question of transitions, some integrality conditions and the moduli space of i_*E (which is also relevant for the duality with the heterotic string). To be somewhat concrete we turn to the well studied example mentioned above, the standard case of Hirzebruch surfaces $B_2 = \mathbf{F}_n$ where some simplifications occur as the first Pontrjagin class $p_1(W) = c_1(W)^2 - 2c_2(W) = 3\sigma(W)$ vanishes. Although it is not important for the conceptual understanding, one may as a minor technical simplification also make the standard assumption that for the gauge bundle $c_1(E)$ vanishes.[3] With all this taken into account, we rewrite once more the change in the tadpole condition for the topological charge of the Yang Mills instanton (including "gravitational" contributions)

$$c_2(E) \quad \longrightarrow \quad c_2(E) - \mathrm{rk}(E)\left(\frac{c_1(N)^2}{8} + \frac{c_1(N)^2}{48}\right). \tag{19}$$

1. Transitions. Note first that this change does not constitute any problem with respect to the seemingly already perfectly "balanced" terms $n_3 + c_2(E)$ in the original equation. The reason is simply that the correction term is only an "once and for all" background term, i.e. the tadpole equation has to be balanced with this term included and then it will not change in transitions as strictly speaking $\mathrm{rk}(E)$ will not change (but of course E will become reducible when the instanton number $c_2(E)$ is lowered in transitions producing points contributing to n_3). So we work in a set-up where the rank $\mathrm{rk}(E)$ of the bundle, which is possibly turned on, is always the maximal one, i.e. $\mathrm{rk}(E) = \mathrm{rk}(G)$ for the gauge group G which comes from the F-theory data and which is eventually partially broken by the embedded bundle E. Note that $\mathrm{rk}(G)$ is the multiplicity k_W of the surface W in the discriminant divisor, resp. the number of seven-branes which have coalesced.

2. Integrality properties. There are some subtle integrality issues pertaining to this formula. The charge (8) and thus the modification of the tadpole condition are not in general in integral cohomology, and thus one may wonder about the integrality prop-

[3] Note however that this is possible only on the $\mathbf{F}_n$ with even n since when W is not a spin manifold $c_1(E)$ has to be odd in order to define a $Spin^c$ structure [20]

erties and the consistency of the tadpole condition. Indeed there are two correction terms in (19), whose integrality properties we would like to understand: the first one $\mathrm{rk}(E)\frac{c_1(N)^2}{8}$ comes just from the term $e^{-c_1(N)/2}$ in (10); the second one $\mathrm{rk}(E)\frac{c_1(N)^2}{48}$ comes from the term $1/\sqrt{\hat{A}(N)}$ in (10).

We start by recalling that the l.h.s. of the tadpole equation (18) is in $\frac{1}{4}\mathbb{Z}$ as the Euler number of X^4 is divisible by 6 [5]. On the other hand the r.h.s. is — as far as the flux term is concerned — in $\frac{1}{4}\mathbb{Z}$ too because (although G can be in the half-integral cohomology as its quantization law [21] demands that $G - c_2(X^4)/2$ lies in the integral cohomology) one finds[4] that $\frac{1}{2}\int_X G \wedge G$, which would seem to lie in $\frac{1}{8}\mathbb{Z}$, also lies in $\frac{1}{4}\mathbb{Z}$. Actually the proof of this fact (cf. the last footnote) shows more, namely that even in the general case of non-integral $\chi(X^4)/24$ nevertheless an half-integral number for the three-brane/bundle contributions is predicted by having G included: for $\chi(X^4)/24 - \frac{1}{2}\int_X G \wedge G$ is congruent to an integer mod $\frac{1}{2}\mathbb{Z}$, i.e. is itself in $\frac{1}{2}\mathbb{Z}$:

$$\frac{\chi(X^4)}{24} - \frac{1}{2}\int_X G \wedge G \in \frac{1}{2}\mathbb{Z}\,. \tag{20}$$

How is this occurence of half-integrality to be explained? Clearly n_3 and $c_2(E)$ should be integral. However we will see that the gravitational contributions (which are related to the geometry of the (compact part W of the) seven-brane, respectively its embedding) come out only half-integral. More precisely we will see that the first correction term accounts for the non integrality (but is still half-integral) whereas the second one is actually integral (at least in the case under consideration $W = \mathbf{F}_n$, and is half-integral in general).

So let us after this prediction (20) from the tadpole equation investigate whether the remaining terms, especially the new ones satisfy this integrality requirement. The base B_3 of the type-IIB theory is assumed to be a spin manifold. We find from $c_1(B_3) \equiv 0 \pmod 2$ and from $c_1(B_3) = c_1 + 2r + t$, computed from the adjunction formula (see footnote 4), that $t \equiv c_1 \pmod 2$ and so $t^2 \equiv c_1^2 \pmod 4$; since $c_1^2 = 8$ for Hirzebruch surfaces[5] one has $t^2 \equiv 0 \pmod 4$ and so the first correction term $\mathrm{rk}(E)t^2/8$ is in $\frac{1}{2}\mathbb{Z}$ (even in the case of $c_1(E) \neq 0$). So the first correction term "explains" the non integrality but does not lead to further problems as it is half-integral (but see the last footnote).

[4] Note that the argument cannot use evenness of the intersection form as this is given only for $c_2(X^4)$ even, which according to the quantization law is exactly not the critical case; instead one argues from $G \equiv c_2(X^4)/2 \pmod{H^4(X^4,\mathbb{Z})}$ that $\frac{1}{2}\int_X G \wedge G \equiv \frac{1}{8}\int_X c_2^2(X^4) \pmod{\frac{1}{2}\mathbb{Z}}$; but $\frac{1}{8}\int_X c_2^2(X^4) \in \frac{1}{4}\mathbb{Z}$ as $\int_X c_2^2(X^4) = 480 + \chi(X^4)/3 \in 2\mathbb{Z}$ (cf. [5]).

[5] For more general B_2 as del Pezzo surfaces dP_k (k-fold blow-up of P^2) or Enriques surface $\mathcal{E}$ of $c_1^2(dP_k) = 9 - k$ resp. $c_1^2(\mathcal{E}) = 0$ the condition $c_1^2 \equiv 0 \pmod 4$ shows that only the bases $dP_1 = F_1, dP_5$, dP_9 and Enriques are unproblematical.

On the other hand the second correction term clearly seems to destroy the demanded (half-)integrality properties in general. To understand this, we have to turn now to the global consistency conditions. So far we have merely focused on the deviation from the simple result $c_2(E)$ for the bundle/instanton contribution for one given bundle. However as already remarked there are necessarily groups of non-parallel seven-branes, and thus the contributions from all of them have to be taken into account. In particular, in the simplest example one has in B_3 at least the surface D_1 of I_1 singularities as further component of the discriminant surface. We recall that since the total space X_4 is a Calabi-Yau manifold, there is a global restriction on the seven-branes, namely the Kodaira condition $\frac{1}{12}\sum_{W_A} k_A r_{W_A} = c_1(B_3)$ (where we sum over all discriminant components W_A of cohomology class r_A and multiplicity k_A) (cf. [2]). Using (17) and[6] $\mathrm{rk}(E_A) = k_A$, this gives exactly the condition necessary for understanding the needed integrality properties:

$$p_1(B_3)\frac{1}{48}\sum_{W_A}\mathrm{rk}(E_A)r_{W_A} = \frac{1}{4}p_1(B_3)c_1(B_3) \tag{21}$$

(which could be even further evaluated as $\frac{1}{2}p_1(B_3)r = \frac{1}{2}(p_1(B_2)+t^2)$). Since B_3 is a spin manifold, i.e. of even c_1, its $p_1 = c_1^2 - 2c_2$ is even too, leading to two factors of 2. Thereby this contribution is actually integral.[7]

3. The moduli space. As we have exchanged the bundle E by the object i_*E when using the "three-dimensional" expression $[\mathrm{ch}(i_*E)\sqrt{\hat{A}(B)}]_3$ instead of the instanton contribution let us finally consider the influence on the moduli space of this shift from E to i_*E. We will find that the dimension of the associated moduli space is unchanged as required by the duality with heterotic string [3]. Whereas in [3] the dimension of the moduli space of E over W was computed "intrinsically" (only with respect to W) and was given by the dimension of $H^1(W, \mathrm{End}(E))$, we have to consider now the torsion sheaf i_*E which lives on B_3. The dimension of the associated moduli space is given by the dimension of $\mathrm{Ext}^1_{B_3}(i_*E, i_*E)$. One can in general expect that the dimension of the moduli space associated to i_*E is bigger than the dimension of the moduli space of E over W. This can happen because, naively speaking, W can move inside B_3 and we therefore have additional deformations. The number of deformations of W in B_3 is simply given by the number of sections of the normal

[6]Strictly speaking we argued for this only for D-branes; however for more general (p,q) seven-branes the couplings relevant for our investigation should be still the same by $\mathrm{Sl}(2,\mathbb{Z})$ duality. By virtue of this duality we impose the "maximal rank condition" $\mathrm{rk}(E_A) = k_A$ also for the other seven-branes (even if these do not contribute to the resulting four-dimensional gauge fields).

[7]Since we have used $c_1^2(N) = p_1(B_3)|_W$, this argument may seem somewhat special for the case of $\mathbf{F}_n$. In general, for $W_A \neq B_2 = \mathbf{F}_n$, and thus $p_1(W) \neq 0$, one may still worry about the integrality properties of $\mathrm{ch}_3(i_*E)$. However by yet another rewriting using the adjunction formula one can isolate a part $\sum_{W_A}\mathrm{rk}(E_A)r_{W_A}\wedge p_1(B)/24$ with the remaining part being a sum of an (integral) index $\int_W \hat{A}(W)e^{c_1(W)/2}\,\mathrm{ch}(E)$ and half-integral terms (using $c_1(B)$ even for B spin).

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bundle, i.e. the dimension of $H^0(N)$. This naive picture can be made precise by considering the long exact sequence (first written down and proven in [22])[8]

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1_W(E,E) \longrightarrow \mathrm{Ext}^1_{B_3}(i_*E, i_*E) \longrightarrow \mathrm{Ext}^0_W(E, E\otimes N) \\ \longrightarrow \mathrm{Ext}^2_W(E,E) \longrightarrow . \end{aligned} \tag{22}$$

Since E is assumed to be a vector bundle over W we have isomorphisms $\mathrm{Ext}^1_W(E,E) = H^1(W, \mathrm{End}(E)), \mathrm{Ext}^0_W(E, E\otimes N) = \mathrm{Hom}(E, E\otimes N)$ and $\mathrm{Ext}^2_W(E,E) = H^2(W, \mathrm{End}(E))$ leading to

$$\begin{aligned} 0 \longrightarrow H^1(W, \mathrm{End}(E)) \longrightarrow \mathrm{Ext}^1_{B_3}(i_*E, i_*E) \longrightarrow \mathrm{Hom}(E, E\otimes N) \\ \longrightarrow H^2(W, \mathrm{End}(E)) \longrightarrow \end{aligned} \tag{23}$$

using the fact that $H^i(W, \mathrm{End}(i_*E)) = H^i(W, i_* \mathrm{End}(E))$ and that $H^i(W, i_* \mathrm{End}(E)) = H^i(W, \mathrm{End}(E))$ [23].

One assumes in general (cf. [3]) that E is a *good* instanton bundle so that $H^2(\mathrm{End}(E)) = 0$ which can be thought as a condition to get a smooth moduli space whose dimension can be evaluated by [24]. One can follow the conditions under which $H^2(W, \mathrm{End}(E))$ vanishes; for this we decompose $H^2(\mathrm{End}(E))$ into its trace-free part $H^2(\mathrm{End}(E)_0)$ and in $h^{(0,2)}(B_2)$. Since we consider rational B_2's we have $h^{(0,2)}(B_2) = 0$ so we are left with the trace-free part. If we assume a high enough instanton number of E, a theorem of Donaldson [25] states that this term vanishes too and we can consider the exact sequence

$$0 \longrightarrow H^1(W, \mathrm{End}(E)) \longrightarrow \mathrm{Ext}^1_{B_3}(i_*E, i_*E) \longrightarrow \mathrm{Hom}(E, E\otimes N) \longrightarrow 0 \tag{24}$$

which gives

$$\dim \mathrm{Ext}^1_{B_3}(i_*E, i_*E) = \dim H^1(W, \mathrm{End}(E)) + \dim H^0(\mathrm{Hom}(E, E\otimes N)) . \tag{25}$$

Now one has $H^0(\mathrm{Hom}(E, E\otimes N)) = H^0(\mathrm{Hom}(E,E)) \otimes H^0(N)$ but for our normal bundle of $c_1(N) = -t$ we have [9] $\mathrm{Hom}(E, E\otimes N) = 0$. So we finally find that the dimensions of the moduli spaces of i_*E and E match.

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[9]As (assuming t effective) any non-trivial element of $H^0(N)$ would give, multiplied by some non-constant section (having zeroes) of N^{-1} (which exists as $c_1(N^{-1}) = t$), a non-constant element of $H^0(\mathbf{C}) = \mathbf{C}$.

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